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Existence and Uniqueness for Nonlinear Boundary Value Problems in Kinetic Theory

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A boundary value problem for the stationary nonlinear Boltzmann equation in a slab has been examined in a weighted L^{∞} space. It has been proved that the problem possesses a unique solution for boundary data small enough. The proof is based on the implicit function theorem. It has also been shown that for the linearized problem the Fredholm alternative applies.

KEY WORDS: Boltzmann equation; boundary value problem; kinetic theory.

1. INTRODUCTION

The development of an existence and uniqueness theorem for a given equation serves several purposes: it helps to recognize whether the solution of a specific problem (either exact or approximate) is representative of more general cases; it may even determine whether a solution exists or not and so provide or remove support to the physical argument that suggests the validity of the equation. The Boltzmann equation is no exception in this respect. Furthermore, there is a widespread belief that any new existence result may throw light on the derivation of the Boltzmann equation itself from first principles.[1,2]

There exists a rather complete theory of initial value problems for the Boltzmann equation in the space-homogeneous case, due to the pioneering papers of Carleman,[3] Morgenstern,[4] and Arkeryd.[5] The situation is far less encouraging when the space dependence is brought in. In fact, no global existence result is available in that case for the initial value problem without severe restrictions on the initial data. If a suitable smallness con-

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dition is included, then several results are available, such as those of Ukai,[6] Nishida and Imai,[7] Shizuta and Asano,[8] and Shizuta[9] for the case of a gas near equilibrium and of Illner and Shinbrot,[10] and Bellomo and Toscani[11] for a gas near vacuum in an infinite space.

Far less is known in the steady case. Ukai and Asano[12] treated the flow past a body in the case of a gas near equilibrium and succeeded in three dimensions but not in two dimensions. This is due to the fact, pointed out by Cercignani,[13] that the Stokes paradox holds in kinetic theory, and accordingly no perturbation about an equilibrium state at rest with the body can give positive results in two dimensions.

An earlier result was given by Pao[14] in the one-dimensional steady case. He started with an existence and uniqueness theorem proved by Cercignani[15] in the linearized case and was able to prove existence for solutions near equilibrium. Since the result of Pao, although quoted in Refs. 16 and 17, seems to have been ignored in the recent literature, and also because some details of his proof may appear insufficient, we give in this paper an alternative proof of Pao's result, i.e., we prove existence for the Boltzmann equation in a slab when the data are close to equilibrium in a weighted L^∞ norm.

2. EXISTENCE OF STATIONARY SOLUTIONS

Our aim is to solve the following weakly nonlinear slab problem:

$$\xi_x \frac{\partial f}{\partial x} = Lf + \nu \Gamma(f, f), \qquad x \in [-1, 1], \quad \xi \in R^3 \tag{1}$$

$$f = g_+, \qquad x = -1, \qquad \xi_x > 0 \tag{2}$$

$$f = g_-, \qquad x = 1, \qquad \xi_x < 0 \tag{3}$$

To simplify notation, we assume that there is a function g such that

$$g = \begin{cases} g_+, & \xi_x > 0 \\ g_-, & \xi_x < 0 \end{cases}$$

Then $g(\xi)$ is defined for all $\xi \in R^3$ and the boundary conditions (2) and (3) can be written as

$$f(-1, \xi) = g(\xi), \qquad \xi_x > 0 \tag{2'}$$

$$f(1, \xi) = g(\xi), \qquad \xi_x < 0 \tag{3'}$$

We state now the properties of the operators L and Γ that will be used throughout the paper. The operator L, which acts only on the ξ dependence of f, can be decomposed as follows:

$$L = -\nu + K \tag{4}$$

where ν is an operator of multiplication by $\nu(\xi)$ and K is an integral operator. The function $\nu(\xi)$ satisfies

$$\nu_0(1 + |\xi|)^\gamma \leqslant \nu(\xi) \leqslant \nu_1(1 + |\xi|)^\gamma \tag{5}$$

where γ depends on an intermolecular potential and $0 < \gamma \leqslant 1$. The operator L is self-adjoint and nonpositive in $L^2(R^3_\xi)$ and possesses a five-dimensional null space $N(L)$. If we decompose $f = q + w$, $q \in N(L)$ and $w \in N(L)^\perp$, then

$$(f, Lf) \leqslant -\mu(w, w), \qquad \mu > 0 \tag{6}$$

To describe properties of the operator $\Gamma(f, g)$, we define the following functional spaces:

$$E_r = \{f \in L^\infty(R^3)\colon (1 + |\xi|^2)^{r/2}\, |f(\xi)| \in L^\infty(R^3)\} \tag{7}$$

with the norm

$$\| f \|_r = \operatorname*{essup}_{\xi \in R^3}\, (1 + |\xi|^2)^{r/2}\, |f(\xi)| \tag{8}$$

The fundamental property of Γ is then

$$\|\Gamma(f, g)\|_r \leqslant c\, \| f \|_r\, \| g\|_r, \qquad r \geqslant 1 \tag{9}$$

Equation (1) with boundary conditions (2′) and (3′) can be transformed to an integral equation. To this end, let us define

$$Uf(x, \xi) = \begin{cases} |\xi_x|^{-1} \displaystyle\int_{-1}^{x} \{\exp[-(x-y)\nu/|\xi_x|]\}\, f(y, \xi)\, dy, & \xi_x > 0 \\ |\xi_x|^{-1} \displaystyle\int_{x}^{1} \{\exp[-(y-x)\nu/|\xi_x|]\}\, f(y, \xi)\, dy, & \xi_x < 0 \end{cases} \tag{10}$$

$$(U_0\, g)(x, \xi) = \begin{cases} g_+ \exp[-(1+x)\nu/|\xi_x|], & \xi_x > 0 \\ g_- \exp[-(1-x)\nu/|\xi_x|], & \xi_x < 0 \end{cases} \tag{11}$$

Then (1) can be replaced by

$$f = U_0\, g + UKf + U\nu\Gamma(f, f) \tag{12}$$

Following Pao,[14] we will solve this integral equation by the implicit function theorem. The solution will be constructed in the space $E = L^\infty([-1, 1], E_2)$. Let us define the following nonlinear operator:

$$G(f, g) = f - U_0 g - UKf - Uv\Gamma(f, f) \tag{13}$$

From definitions (10) and (11), estimate (9), and properties of K, which will be stated in the next section, it easily follows that G is well defined,

$$G\colon \quad E \times E_2 \to E$$

and is continuous with respect to both variables.

Calculating the Frechet differential of G, we obtain

$$G_f(f, g)h = h - UKh - 2Uv\Gamma(f, h)$$

By virtue of (9), $G_f(f, g)$ is a continuous function, as is $G_g(f, g)$. Since $G(0, 0) = 0$, to apply the implicit function theorem it is enough to show that

$$G_f(0, 0)^{-1}\colon \quad E \to E$$

exists and is continuous.

In his paper Pao[14] assumed the existence of G_f^{-1} on the basis of linear theory.[15] However, from results of Ref. 15 the boundedness of $(I - U_1 K_1)^{-1}$ follows, but only for U_1 and K_1, which are essential modifications of U and K, and it is not obvious that the boundedness of $(I - UK)^{-1}$ can be derived from that result.

Main Lemma. The operator $(UK)^4$ is a compact operator in E.

We postpone the proof of this lemma to the next section and derive the existence of $G_f(0, 0)^{-1}$ from it. By the lemma, one can apply to the operator $G_f(0, 0) = I - UK$ the Fredholm alternative. Then it is enough to show that

$$(I - UK)h = 0 \tag{14}$$

has the unique solution $h = 0$.

By (14) and properties of U, the trace of h on a boundary is well defined and we can write (14) in the differential form

$$\xi_x \, \partial h/\partial x = Lh \tag{15}$$

$$h = 0, \qquad x = -1, \qquad \xi_x > 0 \qquad \text{and} \qquad x = 1, \quad \xi_x < 0 \tag{16}$$

We shall show that the only solution of the above boundary value problem in $L^2[-1, 1], L^2(R^3))$ is $h=0$. Since $E \subset L^2[-1, 1], L^2(R^3))$, this guarantees the uniqueness in E as well.

Multiply (15) by h and integrate over x and ξ, using the boundary conditions (16), to obtain

$$-\int_{\xi_x<0} \xi_x h^2(-1, \xi)\, d\xi + \int_{\xi_x>0} \xi_x h^2(1, \xi)\, d\xi = \int_{-1}^{1} dx \int_{R^3} d\xi\, hLh \quad (17)$$

By (6) we have

$$-\int_{\xi_x<0} \xi_x h^2(-1, \xi)\, d\xi + \int_{\xi_x>0} \xi_x h^2(1, \xi)\, d\xi$$
$$+\mu \int_{-1}^{1} dx \int_{R^3} d\xi\, w_h^2 \leqslant 0, \qquad w_h \in N(L)^{\perp} \quad (18)$$

Since all terms in (18) are nonnegative, they must all be zero. Then $w_h=0$ and h is a combination of collision invariants with constant coefficients. But $h(-1, \xi)=h(1, \xi)\equiv 0$, hence $h\equiv 0$. Thus, we have proved the following theorem:

Theorem. The boundary value problem (1), (2′), (3′) possesses in E a unique solution provided the boundary data g have small enough norm in E_2.

3. PROOF OF THE MAIN LEMMA

First, our original problem will be transformed to $L^\infty[-1, 1], L^\infty(R^3))$. To this end, recall that the integral operator K is given by

$$Kf(\xi) = \int_{R^3} k(\xi, \xi_1)\, f(\xi_1)\, d\xi_1$$

Then

$$K_E f(\xi) = \int_{R^3} k_E(\xi, \xi_1)\, f(\xi_1)\, d\xi_1$$

with

$$k_E(\xi, \xi_1) = k(\xi, \xi_1)(1+|\xi|^2)(1+|\xi_1|^2)^{-1}$$

has the following property: If

$$g(x, \xi) = (1+|\xi|^2)\, f(x, \xi) \in L^\infty([-1, 1], L^\infty(R^3))$$

then

$$K_E g = K_E(1+|\xi|^2) f = (1+|\xi|^2)\, Kf$$

We also have

$$U(1+|\xi|^2) f = (1+|\xi|^2)\, Uf$$

Hence, compactness of $(UK)^4$ in E is equivalent to compactness of $(UK_E)^4$ in $L^\infty([-1,1], L^\infty(R^3))$.

Let us consider operators U^* and K_E^* in $L^1([-1,1], L^1(R^3))$ to which U and K_E are adjoint. By Schauder's theorem it is sufficient to prove compactness of $(K_E^* U^*)^4$ in $L^1([-1,1], L^1(R^3))$. By simple calculations we obtain

$$U^* f(x,\xi) = \begin{cases} |\xi_x|^{-1} \displaystyle\int_x^1 \{\exp[-(y-x)\nu/|\xi_x|]\}\, f(y,\xi)\, dy, & \xi_x > 0 \\ |\xi_x|^{-1} \displaystyle\int_{-1}^x \{\exp[-(x-y)\nu/|\xi_x|]\}\, f(y,\xi)\, dy, & \xi_x < 0 \end{cases} \tag{19}$$

$$\begin{aligned} K_E^* f(x,\xi) &= \int_{R^3} k_E^*(\xi,\xi_1)\, f(x,\xi_1)\, d\xi_1 \\ k_E^*(\xi,\xi_1) &= k(\xi,\xi_1)(1+|\xi_1|^2)(1+|\xi|^2)^{-1} \end{aligned} \tag{20}$$

For the kernel $k(\xi,\xi_1)$ the following estimate holds[18]:

$$|k(\xi,\xi_1)| \leqslant c\, |\xi-\xi_1|^{-1} \exp\left[-\alpha\, |\xi-\xi_1|^2 - \alpha\, \frac{(\xi^2-\xi_1^2)^2}{|\xi-\xi_1|^2}\right], \qquad \alpha > 0 \tag{21}$$

It is easy to show that (21) is valid for $k_E(\xi,\xi_1)$ and $k_E^*(\xi,\xi_1)$.

In our proof we will follow the lines of Ref. 19, using the Dunford–Pettis criterion of weak compactness in L^1. To this end let us observe that U^* and K_E^* are bounded operators in $L^1([-1,1], L^1(R^3))$. We will show that $K_E^* U^* K_E^*$ is weakly relatively compact in this space.

Let us put

$$\begin{aligned} I_A = \int_A dx\, d\xi \int_{R^3} d\xi_2\, k_E^*(\xi,\xi_2) \\ \times \int_x^1 dr\, |\xi_{2,x}|^{-1} \exp[-(r-x)\nu/|\xi_{2,x}|] \int_{R^3} d\xi_1\, k_E^*(\xi_2,\xi_1)\, f(r,\xi_1) \end{aligned}$$

which is the part of $\int_A dx\, d\xi\, K_E^* U^* K_E^* f$ for $\xi_{2,x} > 0$.

By (21) we obtain

$$|I_A| \leqslant \int_A dx\, d\xi \int_{R^3} d\xi_2\, |\xi - \xi_2|^{-1} \exp(-\alpha\, |\xi - \xi_2|^2)$$

$$\times \int_x^1 dr\, |\xi_{2,x}|^{-1} \exp[-(r-x)v/|\xi_{2,x}|] \int_{R^3} d\xi_1\, |\xi_2 - \xi_1|^{-1}$$

$$\times\ [\exp(-\alpha\, |\xi_2 - \xi_1|^2)]\, |f(r, \xi_1)|$$

Changing variables $x \to s = r - x$, $\xi \to z = \xi - \xi_2$, we can write

$$|I_A| \leqslant \int_{R^3} d\xi_2 \int_{-1}^1 dr \int_{R^3} d\xi_1\, |\xi_2 - \xi_1|^{-1}[\exp(-\alpha\, |\xi_2 - \xi_1|^2)]\, |f(r, \xi_1)|$$

$$\times \int_{A(r,\xi_2)} z^{-1}[\exp(-\alpha z^2)]\, |\xi_{2,x}|^{-1} \exp(-sv_0/|\xi_{2,x}|)\, dz\, ds$$

where

$$A(r, \xi_2) = \{(s, z)\colon s = r - x,\ s \geqslant 0,\ z = \xi - \xi_2,\ (x, \xi) \in A\}$$

We now want to show that by choosing the measure of the set A small enough, we can make $|I_A|/\|f\|$ as small as we please. To this end, we split the integral defining $|I_A|$ into two parts, I_1 and I_2, referring to $\xi_{2,x} < \eta$ and $> \eta$, respectively. Let us consider I_1 first. Since

$$\int_{A(r,\xi_2)} z^{-1}[\exp(-\alpha z^2)]\, |\xi_{2,x}|^{-1} \exp(-sv_0/|\xi_{2,x}|)\, dz\, ds < c$$

with c independent of $|\xi_{2,x}|$, we have, if

$$\Omega^{\pm} = \{\xi_2\colon |\xi_{2,x}| < \eta,\ |\xi_2 - \xi_1|^{\pm 1} \geqslant \gamma^{\pm 1}\}$$

that

$$I_1 \leqslant c \int_{R^3} d\xi_1 \int_{-1}^1 dr \int_{\Omega^+} d\xi_2\, |\xi_2 - \xi_1|^{-1}\, |f(r, \xi_1)|$$

$$+ \int_{R^3} d\xi \int_{-1}^1 dr \int_{\Omega^-} d\xi\, [\exp(-\alpha\, |\xi_2 - \xi_1|^2)]\, |f(r, \xi_1)|$$

$$\leqslant c(\gamma^2 + \eta\gamma^{-1})\, \|f\|$$

To estimate I_2, we remark that since the set $A(r, \xi_2)$ is obtained by translation of A, then $\mu(A(r, \xi_2)) = \mu(A)$. Hence, for a given $\varepsilon > 0$, we can choose $\delta > 0$ such that if $\mu(A) < \delta$, then

$$\int_{A(r,\xi_2)} z^{-1}[\exp(-\alpha z^2)]\, |\xi_{2,x}|^{-1} \exp(-sv_0/|\xi_{2,x}|)\, dz\, ds < \varepsilon\eta^{-1}$$

Thus,

$$I_1 = \int_{|\xi_{2,x}| > \eta} d\xi_2 \int_{-1}^{1} dr \int_{R^3} d\xi_1 \, |\xi_2 - \xi_1|^{-1} [\exp(-\alpha |\xi_2 - \xi_1|^2)] \, |f(r, \xi_1)|$$
$$\times \int_{A(r,\xi_2)} z^{-1} [\exp(-\alpha z^2)] \, |\xi_{2,x}|^{-1} \exp(-s v_0 / |\xi_{2,x}|) \, dz \, ds$$
$$\leqslant c\varepsilon\eta^{-1} \, \| f \|$$

and

$$|I_A| = I_1 + I_2 \leqslant c(\gamma^2 + \eta\gamma^{-1} + \varepsilon\eta^{-1}) \leqslant 3c\varepsilon^{2/5} \qquad (\gamma = \varepsilon^{1/5}, \eta = \varepsilon^{3/5})$$

A similar estimate can be obtained for the part of $\int_A dx \, d\xi \, K_E^* U^* K_E^* f$ with $\xi_{2,x} < 0$.

To satisfy all conditions of the Dunford–Pettis criterion, we have to show that for any $\varepsilon > 0$ there exists a compact set C such that

$$\int_{[-1,1] \times R^3 \setminus C} dx \, d\xi \, |K_E^* U^* K_E^* f| < \varepsilon \, \| f \|$$

Let $C = [-a, a] \times \{\varnothing\}$, where $0 < 1 - a < \varepsilon$ ($\varnothing$ is the zero vector in R^3). Denoting $[-1, 1] \times R^3 \setminus C = A$, we can repeat the previous estimates. This time, however,

$$\int_{A(r,\xi_2)} z^{-1} [\exp(-\alpha z^2)] \, |\xi_{2,x}|^{-1} [\exp(-s v_0 / |\xi_{2,x}|)] \, dz \, ds$$
$$= \int_{R^3} z^{-1} [\exp(-\alpha z^2)] \, dz \int_{\tilde{A}(r)} |\xi_{2,x}|^{-1} [\exp(-s v_0 / |\xi_{2,x}|)] \, ds < c\varepsilon$$

where

$$\tilde{A}(r) = \{s \colon s = r - x, s \geqslant 0, x \in [-1, -a] \cup [a, 1]\}$$

and $\mu(\tilde{A}(r)) \leqslant 2\varepsilon$. Hence $K_E^* U^* K_E^*$ is weakly relatively compact in $L^1([-1, 1], L^1(R^3))$. The same is true for $(K_E^* U^*)^2$ as a product of weakly compact and bounded operators. Then $(K_E^* U^*)^4$ is compact as a product of weakly compact operators.

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